chapter six

Elasticity

Elasticity

A bridge, when used by traffic during the day, is subjected to loads of varying magnitude. Before a steel bridge is erected, therefore, samples of the steel are sent to a research laboratory, where they undergo tests to find out whether the steel can withstand the loads to which it is likely to be subjected.

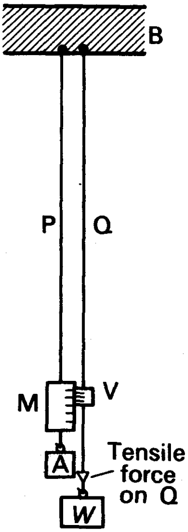


FIG. 6.1
Tensile force

Fig. 6.1 illustrates a simple laboratory method of discovering useful information about the property of steel we are discussing. Two long thin steel wires, P, Q, are suspended beside each other from a rigid support B, such as a girder at the top of the ceiling. The wire P is kept taut by a weight A attached to its end and carries a scale M graduated in centimetres. The wire Q carries a vernier scale V which is alongside the scale M.

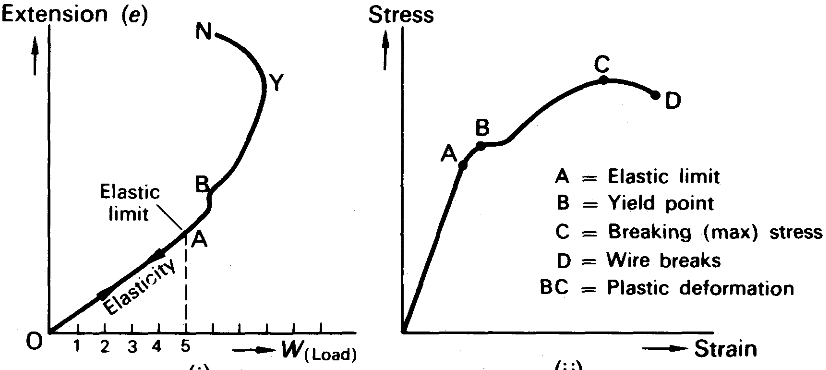
When a load W such as 1 kgf is attached to the end of Q, the wire increases in length by an amount which can be read from the change in the reading on the vernier V. If the load is taken off and the reading on V returns to its original value, the wire is said to be **elastic** for loads from zero to 1 kgf, a term adopted by analogy with an elastic thread. When the load W is increased to 2 kgf the extension (increase in length) is obtained from V again; and if the reading on V returns to origin value when the load is removed the wire is said to be elastic at least for loads from zero to 2 kgf.

The extension of a thin wire such as Q for increasing loads may be found by experiments to be as follows:

W (kgf)	0	1	2	3	4	5	6	7	8
Extension (mm.)	0	0·14	0·28	0·42	0·56	0·70	0·85	1·01	1·19

Elastic Limit

When the extension, e, is plotted against the load, W, a graph is obtained which is a *straight line* OA, followed by a curve ABY rising slowly at first and then very sharply, Fig. 6.2. Up to about 5 kgf, then, the results in the table show that the extension increased by 0·14 mm for every kgf which is added to the wire. Further, the wire returned to its original length when the load was removed. For loads greater than about 5 kgf, however, the extension increases relatively more and more, and the wire now no longer returns to its original length when it is unloaded. The wire is thus permanently strained, and A corresponds to its *elastic limit.*



(i) (ii)

FIG. 6.2 Extension v. Load

Hooke's Law

From the straight line graph OA, we deduce that *the extension is proportional to the load or tension in the wire when the elastic limit is not exceeded.* This is known as *Hooke's law*, after ROBERT HOOKE, founder of the Royal Society, who discovered the relation in 1676. The law shows that when a molecule of a solid is displaced farther from its mean position, the restoring force is proportional to its displacement (see p. 126). One may therefore conclude that the molecules of a solid are undergoing simple harmonic motion (p. 44).

The measurements also show that it would be dangerous to load the wire with weights greater than 5 kilogrammes, the elastic limit, because the wire then suffers a permanent strain. Similar experiments in the research laboratory enable scientists to find the maximum load which a steel bridge, for example, should carry for safety. Rubber samples are also subjected to similar experiments, to find the maximum safe tension in rubber belts used in machinery.

Yield Point. Ductile and Brittle Substances. Breaking Stress

Careful experiments show that, for mild steel and iron for example, the molecules of the wire begin to 'slide' across each other soon after the load exceeds the elastic limit, that is, the material becomes *plastic.* This is indicated by the slight 'kink' at B beyond A in Fig. 6.2 (i), and it is called the *yield point* of the wire. The change from an elastic to a plastic stage is shown by a sudden increase in the extension, and as the load is increased further the extension increases rapidly along the curve YN and the wire then snaps. The *breaking stress* of the wire is the corresponding force per unit area of cross-section of the wire. Substances such as those just described, which elongate considerably and undergo plastic deformation until they break, are known as *ductile* substances. Lead, copper and wrought iron are ductile. Other substances, however, break just after the elastic limit is reached; they are known as *brittle* substances. Glass and high carbon steels are brittle.

Brass, bronze, and many alloys appear to have no yield point. These materials increase in length beyond the elastic limit as the load is increased without the appearance of a plastic stage.

The strength and ductility of a metal, its ability to flow, are dependent on defects in the metal crystal lattice. Such defects may consist of a missing atom at a site or a *dislocation* at a plane of atoms. Plastic deformation is the result of the 'slip' of atomic planes. The latter is due to the movement of dislocations, which spreads across the crystal.

Tensile Stress and Tensile Strain. Young's Modulus

We have now to consider the technical terms used in the subject of elasticity of wires. When a force or tension F is applied to the end of a wire of cross-sectional area A, Fig. 6.3,

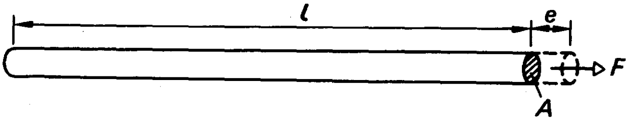


FIG. 6.3 Tensile stress and tensile strain

$$\text{the } \textit{tensile stress} = \textit{force per unit area} = \frac{F}{A} \qquad . \qquad (1)$$

If the extension of the wire is e, and its original length is l,

$$\text{the } \textit{tensile strain} = \textit{extension per unit length} = \frac{e}{l} \qquad . \qquad (2)$$

Suppose 2 kg is attached to the end of a wire of length 2 metres of diameter 0·64 mm, and the extension is 0·60 mm. Then

$$F = 2\,\text{kgf} = 2\times 9{\cdot}8\ \text{N},\ A = \pi\times 0{\cdot}032^2\ \text{cm}^2 = \pi\times 0{\cdot}032^2\times 10^{-4}\ \text{m}^2.$$

$$\therefore\ \text{tensile stress} = \frac{2\times 9{\cdot}8}{\pi\times 0{\cdot}032^2\times 10^{-4}}\ \text{N m}^{-2},$$

and

$$\text{tensile strain} = \frac{0{\cdot}6\times 10^{-3}\ \text{metre}}{2\ \text{metre}} = 0{\cdot}3\times 10^{-3}.$$

It will be noted that 'stress' has units such as 'newton m^{-2}'; 'strain' has no units because it is the ratio of two lengths.

A *modulus of elasticity* of the wire, called **Young's modulus (E)**, is defined as the ratio

$$E = \frac{\text{tensile stress}}{\text{tensile strain}} \quad . \quad . \quad . \quad (3)$$

Thus

$$E = \frac{F/A}{e/l}.$$

Using the above figures,

$$E = \frac{2 \times 9{\cdot}8/(\pi \times 0{\cdot}032^2 \times 10^{-4})}{0{\cdot}3 \times 10^{-3}},$$

$$= \frac{2 \times 9{\cdot}8}{\pi \times 0{\cdot}032^2 \times 10^{-4} \times 0{\cdot}3 \times 10^{-3}},$$

$$= 2{\cdot}0 \times 10^{11}\,\mathrm{N\,m^{-2}}.$$

It should be noted that Young's modulus, E, is calculated from the ratio stress:strain only when the wire is under 'elastic' conditions, that is, the load does not then exceed the elastic limit (p. 154). Fig. 6.2 (ii) shows the general stress-strain diagram for a ductile material.

Dimensions of Young's Modulus

As stated before, the 'strain' of a wire has no dimensions of mass, length, or time, since, by definition, it is the ratio of two lengths. Now

$$\text{dimensions of stress} = \frac{\text{dimensions of force}}{\text{dimensions of area}}$$

$$= \frac{\mathrm{MLT^{-2}}}{\mathrm{L^2}}$$

$$= \mathrm{ML^{-1}T^{-2}}.$$

∴ dimensions of Young's modulus, E,

$$= \frac{\text{dimensions of stress}}{\text{dimensions of strain}}$$

$$= \mathrm{ML^{-1}T^{-2}}.$$

Determination of Young's Modulus

The magnitude of Young's modulus for a material in the form of a wire can be found with the apparatus illustrated in Fig. 6.1, p. 153, to which the reader should now refer. The following practical points should be specially noted:

(1) The wire is made *thin* so that a moderate load of several kilograms produces a large tensile stress. The wire is also made *long* so that a measurable extension is produced.

(2) The use of two wires, P, Q, of the same material and length, eliminates the correction for (i) the yielding of the support when loads are added to Q, (ii) changes of temperature.

(3) Both wires should be free of kinks, otherwise the increase in length cannot be accurately measured. The wires are straightened by attaching weights to their ends, as shown in Fig. 6.1.

(4) A vernier scale is necessary to measure the extension of the wire since this is always small. The 'original length' of the wire is measured from the top B *to the vernier V* by a ruler, since an error of 1 millimetre is negligible compared with an original length of several metres. For very accurate work, the extension can be measured by using a spirit level between the two wires, and adjusting a vernier screw to restore the spirit level to its original reading after a load is added.

(5) The diameter of the wire must be found by a micrometer screw gauge at several places, and the average value then calculated. The area of cross-section, A, $= \pi r^2$, where r is the radius.

(6) The readings on the vernier are also taken when the load is gradually removed in steps of 1 kilogramme; they should be very nearly the same as the readings on the vernier when the weights were added, showing that the elastic limit was not exceeded. Suppose the reading on V for loads, W, of 1 to 6 kilogramme are a, b, c, d, e, f, as follows:

W (kgf)	1	2	3	4	5	6
Reading on V	a	b	c	d	e	f

The average extension for 3 kilogramme is found by taking the average of $(d-a)$, $(e-b)$, and $(f-c)$. Young's modulus can then be calculated from the relation stress/strain, where the stress $= 3\times 9{\cdot}8/\pi r^2$, and the strain = average extension/original length of wire (p. 155).

Magnitude of Young's Modulus

Mild steel (0·2% carbon) has a Young's modulus value of about $2{\cdot}0\times 10^{11}$ newton m^{-2}, copper has a value about $1{\cdot}2\times 10^{11}$ newton m^{-2}; and brass a value about $1{\cdot}0\times 10^{11}$ newton m^{-2}.

The breaking stress (tenacity) of cast-iron metal is about $1{\cdot}5\times 10^8$ newton m^{-2}; the breaking stress of mild steel metal is about $4{\cdot}5\times 10^8$ newton m^{-2}.

At Royal Ordnance and other Ministry of Supply factories, tensile testing is carried out by placing a sample of the material in a machine known as an *extensometer*, which applies stresses of increasing value along the length of the sample and automatically measures the slight increase in length. When the elastic limit is reached, the pointer on the dial of the machine flickers, and soon after the yield point is reached the sample becomes thin at some point and then breaks. A graph showing the load v. extension is recorded automatically by a moving pen while the sample is undergoing test.

EXAMPLE

Find the maximum load in kgf which may be placed on a steel wire of diameter 0·10 cm if the permitted strain must not exceed $\frac{1}{1000}$ and Young's modulus for steel is $2{\cdot}0\times 10^{11}$ N m^{-2}.

We have $$\frac{\text{max. stress}}{\text{max. strain}} = 2\times 10^{11}.$$

$$\therefore \text{ max. stress} = \tfrac{1}{1000}\times 2\times 10^{11} = 2\times 10^8 \text{ N m}^{-2}.$$

Now area of cross-section in m² $= \dfrac{\pi d^2}{4} = \dfrac{\pi \times 0{\cdot}1^2 \times 10^{-4}}{4}$

and
$$\text{stress} = \frac{\text{load } F}{\text{area}}$$

$$\therefore F = \text{stress} \times \text{area} = 2 \times 10^8 \times \frac{\pi \times 0{\cdot}1^2 \times 10^{-4}}{4} \text{ newton}$$

$$= 157 \text{ newton} = 15{\cdot}7 \text{ kgf (approx.)}.$$

since 10 newtons = 1 kgf (approx.).

Force in Bar Due to Contraction or Expansion

When a bar is heated, and then prevented from contracting as it cools, a considerable force is exerted at the ends of the bar. We can derive a formula for the force if we consider a bar of Young's modulus E, a cross-sectional area A, a linear expansivity of magnitude α, and a decrease in temperature of t°C. Then, if the original length of the bar is l, the decrease in length e if the bar were free to contract $= \alpha lt$.

Now
$$E = \frac{F/A}{e/l}.$$

$$\therefore F = \frac{EAe}{l} = \frac{EA\alpha lt}{l}.$$

$$\therefore F = EA\alpha t.$$

As an illustration, suppose a steel rod of cross-sectional area 2·0 cm² is heated to 100°C, and then prevented from contracting when it is cooled to 10°C. The linear expansivity of steel $= 12 \times 10^{-6}$ K^{-1} and Young's modulus $= 2{\cdot}0 \times 10^{11}$ newton m^{-2}. Then

$$A = 2 \text{ cm}^2 = 2 \times 10^{-4} \text{ m}^2, t = 90 \text{ deg C}.$$

$$\therefore F = EA\alpha t = 2 \times 10^{11} \times 2 \times 10^{-4} \times 12 \times 10^{-6} \times 90 \text{ newton (N)},$$

$$= 43200 \text{ N} = \frac{43200}{9{\cdot}8} \text{ kgf} = 4400 \text{ kgf}.$$

Energy Stored in a Wire

Suppose that a wire has an original length l and is stretched by a length e when a force F is applied at one end. If the elastic limit is not exceeded, the extension is directly proportional to the applied load (p. 154). Consequently the force *in the wire* has increased in magnitude from zero to F, and hence the average force in the wire while stretching was $F/2$. Now

$$\text{work done} = \text{force} \times \text{distance}.$$

$$\therefore \text{work} = \text{average force} \times \text{extension}$$

$$= \tfrac{1}{2}Fe \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the amount of energy stored in the wire. The formula $\frac{1}{2}Fe$ gives the energy in *joule* when F is in newton and e is in metre.

Further, since $F = EAe/l$,

$$\text{energy} = \tfrac{1}{2}EA\frac{e^2}{l}.$$

As an illustration, suppose $E = 2{\cdot}0\times10^{11}$ newton m^{-2}, $A = 3\times10^{-2}$ cm^2 $= 3\times10^{-6}$ m^2, $e = 1$ mm $= 1\times10^{-3}$ m, $l = 400$ cm $=$ 4 m. Then

$$\text{energy stored} = \tfrac{1}{2}EA\frac{e^2}{l} = \tfrac{1}{2}\times\frac{2\times10^{11}\times3\times10^{-6}\times(1\times10^{-3})^2}{4}\text{ joule,}$$

$$= 0{\cdot}075 \text{ J.}$$

The volume of the wire $= Al$. Thus, from (1),

$$\text{energy per unit volume} = \tfrac{1}{2}\frac{Fe}{Al} = \tfrac{1}{2}\frac{F}{A}\times\frac{e}{l}.$$

But F/A = stress, e/l = strain,

$$\therefore \text{ energy per unit volume} = \tfrac{1}{2}\text{ stress}\times\text{strain} \quad . \quad (2)$$

Graph of *F* v. *e* and energy

The energy stored in the wire when it is stretched can also be found from the graph of F v. e. Fig. 6.4. Suppose the wire extension is e_1 when a load F_1 is applied, and the extension increases to e_2 when the load increases to F_2. If F is the load between F_1 and F_2 at some stage, and Δx is the small extension which then occurs, then

$$\text{energy stored} = \text{work done} = F{\cdot}\Delta x.$$

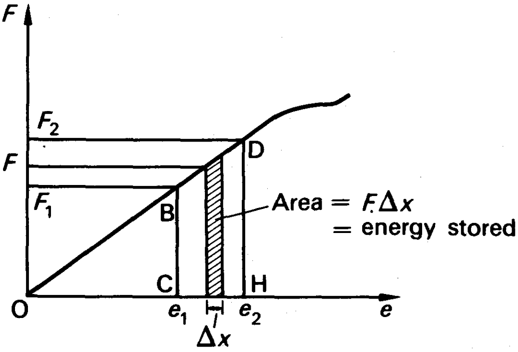


FIG. 6.4 Energy in stretched wire

Now $F.\Delta x$ is represented by the small *area* between the axis of e and the graph, shown shaded in Fig. 6.4. Thus the total work done between e_1 and e_2 is represented by the area CBDH.

If the extension occurs on the straight part of the curve, when Hooke's law is obeyed, then CBDH is a trapezium. The area of a trapezium = half the sum of the parallel sides × perpendicular distance between them $= \frac{1}{2}(\text{BC}+\text{DH}) \times \text{CH} = \frac{1}{2}(F_1+F_2)(e_2-e_1)$.

$\therefore$ *energy stored = average force × increase in length.*

If the extension occurs beyond the elastic limit, for example, along the curved part of the graph in Fig. 6.4, the energy expended can be obtained from the area between the curve and the axis of e.

EXAMPLES

1. A 20 kg weight is suspended from a length of copper wire 1 mm in radius. If the wire breaks suddenly, does its temperature increase or decrease? Calculate the change in temperature; Young's modulus for copper $= 12 \times 10^{10}$ N m^{-2}; density of copper = 9000 kg m^{-3}; specific heat capacity of copper = 0·42 J g^{-1} K^{-1}.) (*C.S.*)

When the wire is stretched, it gains potential energy equal to the work done on it. When the wire is suddenly broken, this potential energy is released as the molecules return to their original position. The energy is converted into heat and thus the temperature rises.

$$\text{Gain in potential energy of molecules} = \text{work done in stretching wire}$$
$$= \tfrac{1}{2}\text{ force } (F) \times \text{extension } (e).$$

With the usual notation, $F = EA\dfrac{e}{l}$

$$\therefore e = \frac{Fl}{E\,.\,A} = \frac{(20 \times 9{\cdot}8) \times l}{12 \times 10^{10} \times \pi \times (10^{-3})^2}\text{ m} = 5{\cdot}2 \times 10^{-4} l\text{ m},$$

$$\therefore \text{potential energy gained} = \tfrac{1}{2} \times 20 \times 9{\cdot}8 \times 5{\cdot}2 \times 10^{-4} l = 5{\cdot}1 \times 10^{-2} l\text{ J}$$

$$\text{Heat capacity of wire} = \text{mass} \times \text{specific heat capacity}$$
$$= \pi \times (10^{-3})^2 \times 9000 l \times (0{\cdot}42 \times 1000) = 11{\cdot}9 l\text{ J K}^{-1}$$

$$\therefore \text{temperature rise} = \frac{\text{potential energy}}{\text{heat capacity}} = \frac{5{\cdot}1 \times 10^{-2} l}{11{\cdot}9 l}$$
$$= 4{\cdot}3 \times 10^{-3}\text{ deg C.}$$

2. Define *stress* and *strain*. Describe the behaviour of a copper wire when it is subjected to an increasing longitudinal stress. Draw a stress-strain diagram and mark on it the elastic region, yield point and breaking stress.

A wire of length 5 m, of uniform circular cross-section of radius 1 mm is extended by 1·5 mm when subjected to a uniform tension of 100 newton. Calculate from first principles the strain energy per unit volume assuming that deformation obeys Hooke's law.

Show how the stress-strain diagram may be used to calculate the work done in producing a given strain, when the material is stretched beyond the Hooke's law region. (*O. & C.*)

$$\text{Strain energy} = \tfrac{1}{2}\text{ tension} \times \text{extension}$$

Tension = 100 newton. Extension $= 1{\cdot}5 \times 10^{-3}$ m.

$$\therefore \text{energy} = \tfrac{1}{2} \times 100 \times 1{\cdot}5 \times 10^{-3} = 0{\cdot}075\text{ J.}$$

Volume of wire = length × area = $5 \times \pi \times 1 \times 10^{-6}$ m³.

$$\therefore \text{ energy per unit volume} = \frac{0{\cdot}075}{5 \times \pi \times 1 \times 10^{-6}} = 4{\cdot}7 \times 10^{3}\ \mathrm{J\,m^{-3}}\ \text{(approx.)}.$$

Bulk Modulus

When a gas or a liquid is subjected to an increased pressure the substance contracts. A change in bulk thus occurs, and the *bulk strain* is defined by:

$$\text{strain} = \frac{\text{change in volume}}{\text{original volume}}.$$

The *bulk stress* on the substance is the increased force per unit area, by definition, and the bulk modulus, K, is given by:

$$K = \frac{\text{bulk stress}}{\text{bulk strain}}$$

$$= \frac{\text{increase in force per unit area}}{\text{change in volume/original volume}}.$$

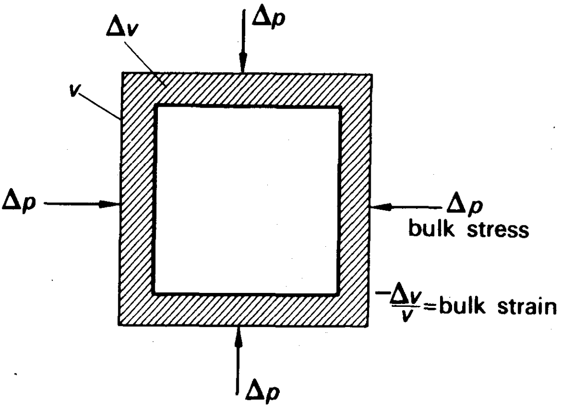


FIG. 6.5 Bulk stress and bulk strain

If the original volume of the substance is v, the change in volume may be denoted by $-\Delta v$ when the pressure increases by a small amount Δp; the minus indicates that the volume decreases. Thus (Fig. 6.5)

$$K = -\frac{\Delta p}{\Delta v/v}.$$

When δp and δv become very small, then, in the limit,

$$K = -v\frac{dp}{dv} \quad . \quad . \quad . \quad . \quad (1)$$

The bulk modulus of water is about 2×10^{9} N m^{-2} for pressures in the range 1 – 25 atmospheres; the bulk modulus of mercury is about

27×10^9 N m^{-2}. The bulk modulus of gases depends on the pressure, as now explained. Generally, since the volume change is relatively large, the bulk modulus of a gas is low compared with that of a liquid.

Bulk Modulus of a Gas

If the pressure, p, and volume, v, of a gas change under conditions such that

$$pv = \text{constant},$$

which is Boyle's law, the changes are said to be *isothermal* ones. In this case, by differentiating the product pv with respect to v, we have

$$p + v\frac{dp}{dv} = 0.$$

$$\therefore\ p = -v\frac{dp}{dv}.$$

But the bulk modulus, K, of the gas is equal to $-v\dfrac{dp}{dv}$ by definition (see p. 161).

$$\therefore\ K = p \qquad . \quad . \quad . \quad . \qquad (2)$$

Thus the *isothermal bulk modulus is equal to the pressure.*

When the pressure, p, and volume, v, of a gas change under conditions such that

$$pv^\gamma = \text{constant},$$

where $\gamma = c_p/c_V$ = the ratio of the specific heat capacities of the gas, the changes are said to be *adiabatic* ones. This equation is the one obeyed by local values of pressure and volume in air when a sound wave travels through it. Differentiating both sides with respect to v,

$$\therefore\ p \times \gamma v^{\gamma-1} + v^\gamma\frac{dp}{dv} = 0,$$

$$\therefore\ \gamma p = -v\frac{dp}{dv},$$

$$\therefore\ \text{adiabatic bulk modulus} = \gamma p \qquad . \quad . \quad . \quad . \quad . \qquad (3)$$

For air at normal pressure, $K = 10^5$ newton m^{-2} isothermally and $1{\cdot}4 \times 10^5$ newton m^{-2} adiabatically. The values of K are of the order 10^5 times smaller than liquids as gases are much more compressible.

Velocity of Sound

The velocity of sound waves through any material depends on (i) its density ρ, (ii) its modulus of elasticity, E. Thus if V is the velocity, we may say that

$$V = kE^x\rho^y \qquad . \quad . \quad . \quad . \qquad \text{(i)},$$

where k is a constant and x, y are indices we can find by the theory of dimensions (p. 34).

The units of velocity, V, are LT^{-1}; the units of density ρ are ML^{-3}; and the units of modulus of elasticity, E, are $ML^{-1}T^{-2}$ (see p. 156). Equating the dimensions on both sides of (i),

$$\therefore\ LT^{-1} = (ML^{-1}T^{-2})^x \times (ML^{-3})^y.$$

Equating the indices of M, L, T on both sides, we have

$$0 = x+y,$$
$$1 = -x-3y,$$
$$-1 = -2x.$$

Solving, we find $x = \frac{1}{2}$, $y = -\frac{1}{2}$. Thus $V = kE^{\frac{1}{2}}\rho^{-\frac{1}{2}}$. A rigid investigation shows $k = 1$, and thus

$$V = E^{\frac{1}{2}}\rho^{-\frac{1}{2}} = \sqrt{\frac{E}{\rho}}.$$

In the case of a solid, E is Young's modulus. In the case of air and other gases, and of liquids, E is replaced by the bulk modulus K. Laplace showed that the adiabatic bulk modulus must be used in the case of a gas, and since this is γp, the velocity of sound in a gas is given by the expression

$$V = \sqrt{\frac{\gamma p}{\rho}}.$$

Modulus of Rigidity

So far we have considered the strain in one direction, or tensile strain, to which Young's modulus is applicable and the strain in bulk or volume, to which the bulk modulus is applicable.

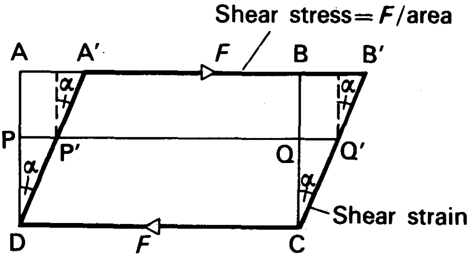


FIG. 6.6 Shear stress and shear strain

Consider a block of material ABCD, such as pitch or plastic for convenience. Fig. 6.6. Suppose the lower plane CD is fixed, and a stress parallel to CD is applied by a force F to the upper side AB. The block then changes its shape and takes up a position A′B′CD. It can now be seen that planes in the material parallel to DC are displaced relative to each other. The plane AB, for example, which was originally directly opposite the plane PQ, is displaced to A′B′ and PQ is displaced to P′Q′. The *angular displacement* α is defined as the *shear strain*. α is the angular displacement between any two planes, for example, between CD and P′Q′.

No volume change occurs in Fig. 6.6. Further, since the force along CD is F in magnitude, it forms a *couple* with the force F applied to the upper side AB. The *shear stress* is defined as the 'shear force per unit area' on the face AB (or CD), as in Young's modulus or the bulk modulus. Unlike the case for these modulii, however, the shear stress has a turning or 'displacement' effect owing to the couple present. The solid does not collapse because in a strained equilibrium position such as A′B′CD in Fig. 6.6, the external couple acting on the solid due to the forces F is balanced by an opposing couple due to stresses inside the material.

If the elastic limit is not exceeded when a shear stress is applied, that is, the solid recovers its original shape when the stress is removed, the *modulus of rigidity*, G, is defined by:

$$G = \frac{\textit{shear stress (force per unit area)}}{\textit{shear strain (angular displacement, } \alpha)}.$$

Shear strain has no units; shear stress has units of newton m^{-2}. The modulus of rigidity of copper is $4{\cdot}8\times10^{10}$ N m^{-2}; for phosphor-bronze it is $4{\cdot}4\times10^{10}$ N m^{-2}, and for quartz fibre it is $3{\cdot}0\times10^{10}$ N m^{-2}.

If a spiral spring is stretched, all parts of the spiral become twisted. The applied force has thus developed a 'torsional' or shear strain. The extension of the spring hence depends on its modulus of rigidity, in addition to its dimensions.

Torsion wire

In sensitive current-measuring instruments, a very weak control is needed for the rotation of the instrument coil. This may be provided by using a long elastic or *torsion wire* of phosphor bronze in place of a spring. The coil is suspended from the lower end of the wire and when it rotates through an angle θ, the wire sets up a weak opposing couple equal to $c\theta$, where c is the elastic constant of the wire. Quartz fibres are very fine but comparitively strong, and have elastic properties. They are also used for sensitive control (see p. 61).

The magnitude of c, the elastic constant, can be derived as follows. Consider a wire of radius a, length l, modulus of rigidity G, fixed at the upper end and twisted by a couple of moment C at the other end. If we take a section of the cylindrical wire between radii r and $r+\delta r$, then a 'slice' of the material ODBX has been sheared through an angle α to a position ODB$_1$X where X is the centre of the lower end of the wire. Fig. 6.7. From the definition of modulus of rigidity, $G =$ torsional stress ÷ torsional strain $= F/A \div \alpha$, where F is the tangential force applied over an area A.

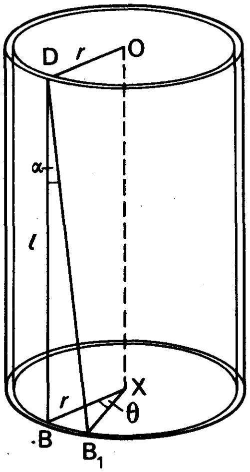


FIG. 6.7
Shear (Torsion) in wire

Now $A =$ area of circular annulus at lower end $= 2\pi r.\delta r$.

$$\therefore\ F = GA\alpha = G.2\pi r.\delta r.\alpha.$$

From Fig. 6.7, it follows that BB$_1$ $= l\alpha$, and BB$_1$ $= r\theta$.

$$\therefore\ l\alpha = r\theta, \text{ or } \alpha = r\theta/l.$$

$$\therefore\ F = \frac{G.2\pi r\ \delta r.r\theta}{l} = \frac{2\pi G\theta r^2.\delta r}{l}.$$

$$\therefore \text{ moment of } F \text{ about axis OX of wire} = F.r$$

$$= \frac{2\pi G\theta}{l}.r^3.\delta r.$$

$$\therefore \text{ total moment, or couple torque } C,$$

$$= \int_0^a \frac{2\pi G\theta}{l}.r^3 dr = \frac{2\pi G\theta}{l}.\frac{a^4}{4}$$

$$\therefore C = \frac{\pi G a^4 \theta}{2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

If the wire is a hollow cylinder of radii a, b respectively, the limits of integration are altered accordingly, and

$$\text{moment of couple} = \int_a^b \frac{2\pi G\theta}{l}.r^3 dr = \frac{\pi G(b^4 - a^4)\theta}{2l}.$$

Determinations of modulus of rigidity. Dynamical method. One method of measuring the modulus of rigidity of a wire E is to clamp it vertically at one end, attach a horizontal disc D of known moment of inertia, I, at the other end, and then time the horizontal torsional oscillations of D. Fig. 6.8 (i). On p. 90, it was shown that the period of oscillation, T, $= 2\pi\sqrt{I/c}$, where c is the opposing couple per unit angle of twist. Thus, with our previous notation, as $\theta = 1$,

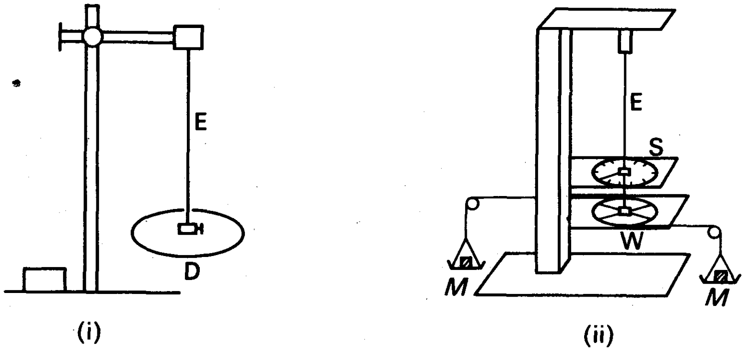
$$c = \frac{\pi G a^4}{2l}.$$

$$\therefore T = 2\pi\sqrt{\frac{2lI}{\pi G a^4}}.$$

or

$$G = \frac{8\pi l I}{a^4 T^2}$$

Hence G can be evaluated from measurements of l, a, I, T.



(i) (ii)

FIG. 6.8 Modulus of rigidity measurement

Statical method. The modulus of rigidity, G, of the wire E can also be found by measuring the steady deflection θ at the lower end on a scale S graduated in degrees when a couple is applied round a wheel W. Fig. 6.8 (ii). If M is the mass in each scale-pan, and d is the diameter of W, the moment of the couple on the wire $= Mgd = \pi G a^4 \theta/2l$. The angle θ in radians, and a, l, are known, and hence G can be evaluated.

Poisson's Ratio

When a rubber cord is extended its diameter usually decreases at the same time. *Poisson's ratio*, σ, is the name given to the ratio

$$\frac{\text{lateral contraction/original diameter}}{\text{longitudinal extension/original length}} \quad . \quad . \quad . \quad (1)$$

and is a constant for a given material. If the original length of a rubber strip is 100 cm and it is stretched to 102 cm, the fractional longitudinal extension = 2/100. If the original diameter of the cord is 0·5 cm and it decreases to 0·495 cm, the fractional lateral contraction = 0·005/0·5 = 1/100. Thus, from the definition of Poisson's ratio,

$$\sigma = \frac{1/100}{2/100} = \tfrac{1}{2}.$$

When the *volume* of a strip of material remains *constant* while an extension and a lateral contraction takes place, it can easily be shown that Poisson's ratio is 0·5 in this case. Thus suppose that the length of the strip is l and the radius is r.

Then $$\text{volume, } V, = \pi r^2 l.$$

By differentiating both sides, noting that V is a constant and that we have a product of variables on the right side,

$$\therefore 0 = \pi r^2 \times \delta l + l \times 2\pi r \delta r.$$

$$\therefore r\delta l = -2l\delta r.$$

$$\therefore -\frac{\delta r/r}{\delta l/l} = \tfrac{1}{2}.$$

But $-\delta r/r$ is the lateral contraction in radius/original radius, and $\delta l/l$ is the longitudinal extension/original length.

$$\therefore \text{Poisson's ratio, } \sigma, = \tfrac{1}{2}.$$

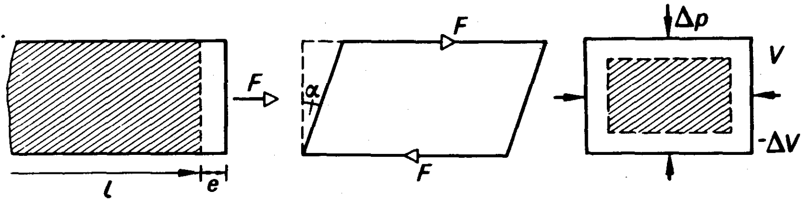
Experiments show that σ is 0·48 for rubber, 0·29 for steel, 0·27 for iron, and 0·26 for copper. Thus the three metals increase in volume when stretched, whereas rubber remains almost unchanged in volume.

Summary

The three modulii of elasticity are compared in the table below:

	Young's modulus, E	Modulus of Rigidity, G	Bulk modulus, K
1.	Definition: $\dfrac{\text{tensile stress}}{\text{tensile strain}}$	$\dfrac{\text{shear stress}}{\text{shear strain}}$	Definition: $\dfrac{\text{pressure change}}{-\Delta v/v}$
2.	Relates to change in *length* ('tensile')	Relates to change in *shape* ('shear')	Relates to change in *volume* ('bulk')

3.



$$E = \frac{F/A}{e/l} \qquad G = \frac{F/A}{\alpha} \qquad K = \frac{\Delta p}{-\Delta v/v}$$

FIG. 6.9

4.	Applies only to solids	Applies to solids and liquids	Applies to all materials. Low value for gases
5.	Used in stretching wires, bending beams, linear expansion and contraction with temperature change	Used in torsion wires, helical springs	Used in velocity of sound formula for all materials. In gas, $K = p$ (isothermal) or γp (adiabatic)

EXERCISES 6

(Assume $g = 9{\cdot}8$ m s^{-2} unless otherwise stated)

What are the missing words in the statements 1–6?

1. When a weight is attached to a suspended long wire, it produces a . . . strain.

2. The units of Young's modulus are . . .

3. In measuring Young's modulus, the . . . must not be exceeded.

4. The energy gained by a wire when stretched = . . . × extension.

5. Bulk stress is defined as the . . . change.

6. When a wire is twisted, a . . . strain is produced.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 9–12?

7. If a metal bar, coefficient of linear expansion α, Young's modulus E, area of cross-section A and length l, is heated through t°C when clamped at both ends, the force in the bar is calculated from *A* $EAlt$, *B* EAt/α, *C* $EA\alpha t$, *D* $E^2A\alpha l$, *E* $A\alpha tl$.

8. When a spiral spring is stretched by a weight attached to it, the strain is *A* tensile, *B* shear, *C* bulk, *D* elastic, *E* plastic.

9. The energy in a stretched wire is *A* $\frac{1}{2}$ load × extension, *B* load × extension, *C* stress × extension, *D* load × strain, *E* $\frac{1}{2}$ load × strain.

10. In an experiment to measure Young's modulus, the wire is thin and long so that *A* very heavy weights can be attached, *B* the wire can be suspended from the ceiling, *C* another identical wire can be arranged parallel to it, *D* the stress is large and the extension is measurable for laboratory loads, *E* a micrometer gauge can be used for accurate measurement.

11. Define *tensile stress*, *tensile strain*, *Young's modulus*. What are the units and dimensions of each?

A load of 2 kgf is applied to the ends of a wire 4 m long, and produces an extension of 0·24 mm. If the diameter of the wire is 2 mm, calculate the stress on the wire, its strain, and the value of Young's modulus.

12. What load in kilogrammes must be applied to a steel wire 6 m long and diameter 1·6 mm to produce an extension of 1 mm? (Young's modulus for steel = $2{\cdot}0 \times 10^{11}$ N m^{-2}.)

13. Find the extension produced in a copper wire of length 2 m and diameter 3 mm when a load of 3 kgf is applied. (Young's modulus for copper = $1{\cdot}1 \times 10^{11}$ N m^{-2}.)

14. What is meant by (i) elastic limit, (ii) Hooke's law, (iii) yield point, (iv) perfectly elastic? Draw sketches of stress *v.* strain to illustrate your answers.

15. '*In an experiment to determine Young's modulus, the strain should not exceed* 1 *in* 1000.' Explain why this limitation is necessary and describe an experiment to determine Young's modulus for the material of a metal wire.

In such an experiment, a brass wire of diameter 0·0950 cm is used. If Young's modulus for brass is $9{\cdot}86 \times 10^{10}$ newton m^{-2}, find in kilogram force the greatest permissible load. (*L.*)

16. Define *stress* and *strain*, and explain why these quantities are useful in studying the elastic behaviour of a material.

State one advantage and one disadvantage in using a long wire rather than a short stout bar when measuring Young's modulus by direct stretching.

Calculate the minimum tension with which platinum wire of diameter 0·1 mm must be mounted between two points in a stout invar frame if the wire is to remain taut when the temperature rises 100K. Platinum has coefficient of linear expansion 9×10^{-6} K^{-1} and Young's modulus 17×10^{10} N m^{-2}. The thermal expansion of invar may be neglected. (*O. & C.*)

17. Explain the terms *stress*, *strain*, *modulus of elasticity* and *elastic limit*. Derive an expression in terms of the tensile force and extension for the energy stored in a stretched rubber cord which obeys Hooke's law.

The rubber cord of a catapult has a cross-sectional area 1·0 mm^2 and a total unstretched length 10·0 cm. It is stretched to 12·0 cm and then released to project a missile of mass 5·0 g. From energy considerations, or otherwise, calculate the velocity of projection, taking Young's modulus for the rubber as $5{\cdot}0 \times 10^{8}$ N m^{-2}. State the assumptions made in your calculation.

18. State Hooke's law, and describe in detail how it may be verified experimentally for copper wire. A copper wire, 200 cm long and 1·22 mm diameter, is fixed horizontally to two rigid supports 200 cm long. Find the mass in grams of the load which, when suspended at the mid-point of the wire, produces a sag of 2 cm at that point. Young's modulus for copper = $12{\cdot}3 \times 10^{10}$ N m^{-2}. (*L.*)

19. Distinguish between Young's modulus, the bulk modulus and the shear modulus of a material. Describe a method for measuring Young's modulus. Discuss the probable sources of error and assess the magnitude of the contribution from each.

A piece of copper wire has twice the radius of a piece of steel wire. Young's modulus for steel is twice that for the copper. One end of the copper wire is joined to one end of the steel wire so that both can be subjected to the same longitudinal force. By what fraction of its length will the steel have stretched when the length of the copper has increased by 1%? (*O. & C.*)

20. In an experiment to measure Young's modulus for steel a wire is suspended vertically and loaded at the free end. In such an experiment, (*a*) why is the wire long and thin, (*b*) why is a second steel wire suspended adjacent to the first?

Sketch the graph you would expect to obtain in such an experiment showing the relation between the applied load and the extension of the wire. Show how it is possible to use the graph to determine (*a*) Young's modulus for the wire, (*b*) the work done in stretching the wire.

If Young's modulus for steel is $2{\cdot}00 \times 10^{11}$ N m^{-2}, calculate the work done in stretching a steel wire 100 cm in length and of cross-sectional area 0·030 cm^2 when a load of 10 kgf is slowly applied without the elastic limit being reached. (*N.*)

21. Describe the changes which take place when a wire is subjected to a steadily increasing tension. Include in your description a sketch graph of tension against extension for (*a*) a ductile material such as drawn copper and (*b*) a brittle one such as cast iron.

Show that the energy stored in a rod of length L when it is extended by a length l is $\frac{1}{2}El^2/L^2$ per unit volume where E is Young's modulus of the material.

A railway track uses long welded steel rails which are prevented from expanding by friction in the clamps. If the cross-sectional area of each rail is 75 cm^2 what is the elastic energy stored per kilometre of track when its temperature is raised by 10°C? (Coefficient of thermal expansion of steel = $1{\cdot}2 \times 10^{-5}$ K^{-1}; Young's modulus for steel = 2×10^{11} N m^{-2}.) (*O. & C.*)

22. What is meant by saying that a substance is 'elastic'?

A vertical brass rod of circular section is loaded by placing a 5 kg weight on top of it. If its length is 50 cm, its radius of cross-section 1 cm, and the Young's modulus of the material $3{\cdot}5 \times 10^{10}$ N m^{-2}, find (*a*) the contraction of the rod, (*b*) the energy stored in it. (*C.*)

23. Give a short account of what happens when a copper wire is stretched under a gradually increasing load. What is meant by *modulus of elasticity, elastic limit, perfectly elastic*?

When a rubber cord is stretched the change in volume is very small compared with the change in shape. What will be the numerical value of Poisson's ratio for rubber, i.e., the ratio of the fractional decrease in diameter of the stretched cord to its fractional increase in length? (*L.*)

24. Describe an accurate method of determining Young's modulus for a wire. Detail the precuations necessary to avoid error, and estimate the accuracy attainable.

A steel tyre is heated and slipped on to a wheel of radius 40 cm which it fits exactly at a temperature t°C. What is the maximum value of t if the tyre is not to be stretched beyond its elastic limit when it has cooled to air temperature (17°C)? What will then be the tension in the tyre, assuming it to be 4 cm wide and 3 mm thick? The value of Young's modulus for steel is $1{\cdot}96 \times 10^{11}$ N m^{-2}, its coefficient of linear expansion is $1{\cdot}1 \times 10^{-5}$ K^{-1}, and its elastic limit occurs for a tension of $2{\cdot}75 \times 10^{8}$ N m^{-2}. The wheel may be assumed to be at air temperature throughout, and to be incompressible. (*O. & C.*)

25. State Hooke's law and describe, with the help of a rough graph, the behaviour of a copper wire which hangs vertically and is loaded with a gradually increasing load until it finally breaks. Describe the effect of gradually reducing the load to zero (*a*) before, (*b*) after the elastic limit has been reached.

A uniform steel wire of density 7800 kg m^{-3} weighs 16 g and is 250 cm long. It lengthens by 1·2 mm when stretched by a force of 8 kgf. Calculate (*a*) the value of Young's modulus for the steel, (*b*) the energy stored in the wire. (*N.*)

26. Describe an experimental method for the determination of (*a*) Young's modulus, (*b*) the elastic limit, of a metal in the form of a thin wire.

A steel rod of mass 97·5 g and of length 50 cm is heated to 200°C and its ends securely clamped. Calculate the tension in the rod when its temperature is reduced to 0°C, explaining how the calculation is made. (Young's modulus for steel = $2{\cdot}0 \times 10^{11}$ N m^{-2}; linear expansivity = $1{\cdot}1 \times 10^{-5}$ K^{-1}; density of steel = 7800 kg m^{-3}.) (*L.*)

27. What do you understand by Hooke's law of elasticity? Describe how you would verify it in any particular case.

A wire of radius 0·2 mm is extended by 0·1% of its length when it supports a load of 1 kg; calculate Young's modulus for the material of the wire. (*L.*)

28. Define Young's modulus of elasticity. Describe an accurate method of determining it. The rubber cord of a catapult is pulled back until its original length has been doubled. Assuming that the cross-section of the cord is 2 mm square, and that Young's modulus for rubber is 10^7 N m^{-2} calculate the tension in the cord. If the two arms of the catapult are 6 cm apart, and the unstretched length of the cord is 8 cm what is the stretching force? (*O. & C.*)

29. Define *Young's modulus of elasticity* and *coefficient of linear expansion.* State units in which each may be expressed and describe an experimental determination of Young's modulus.

For steel. Young's modulus is $1{\cdot}8 \times 10^{11}$ N m^{-2} and the coefficient of expansion $1{\cdot}1 \times 10^{-5}$ K^{-1}. A steel wire 1 mm in diameter is stretched between two supports when its temperature is 200°C. By how much will the force the wire exerts on the supports increase when it cools to 20°C, if they do not yield? Express the answer in terms of the weight of a kilogramme. (*L.*)

30. Define *elastic limit* and *Young's modulus* and describe how you would find the values for a copper wire.

What stress would cause a wire to increase in length by one-tenth of one per cent if Young's modulus for the wire is 12×10^{10} N m^{-2}? What load in kg wt. would produce this stress if the diameter of the wire is 0·56 mm? (*L.*)